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THE NEUTRALIZATION PROBLEM OF IONIC PROPULSION

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ABSTRACT

Electron-ion collisions are an essential consideration in the neutralization problem of ionic propulsion, even for that part of the ionized gas that is much closer to the vehicle than one mean free path. This is because electrons with a history of collisions downstream will diffuse back upstream and will constitute a majority of the electron charge density near the vehicle.

An intuitive and semiquantitative treatment indicates that neutralization is relatively easy to accomplish. Important considerations are to locate the electron-emitting filaments in or as near as possible to the ion stream, and to bias the emitter a few volts positive with respect to the vehicle's external structure. The bias prevents electron loss by assuring that the structures are not accessible to the electrons. The distance of the electron emitter behind the vehicle and the directivity of emissions are not important.

The spatial distribution of ions and electrons is studied in an approximate way. The design of tank experiments is discussed briefly.

I. INTRODUCTION[1]

The feasibility of space vehicle propulsion by means of a cesium ion beam has been doubted because of the belief that the positive space charge aft of the vehicle may not be readily neutralized by electrons. The purpose of this report is to show that

[1]This paper presents the results of one phase of research carried out at the Jet Propulsion Laboratory, California Institute of Technology, under Contract No. NASw-6, sponsored by the National Aeronautics and Space Administration.

there is actually very little difficulty in neutralization. In addition, neutralizer design is discussed together with the design of tank experiments which adequately represent space environment. The approach here is intuitive and approximate. Others are engaged in detailed calculations of ion and electron distributions; no attempt is made to duplicate their work. Rather, the purpose is to develop a feel for the behavior of the ion - electron mixture that will serve as a guide to experimental design and to selection of realistic boundary conditions for detailed calculations.

II. DESCRIPTION

In Fig. 1, the ion accelerator is represented schematically as three electrodes. These are, from left to right, an emitter, an accelerating grid, and a decelerating grid (emit, accel, decel). Accelerator problems are well understood and subject to experimental test. The concern here is with the beam after it leaves the last grid.

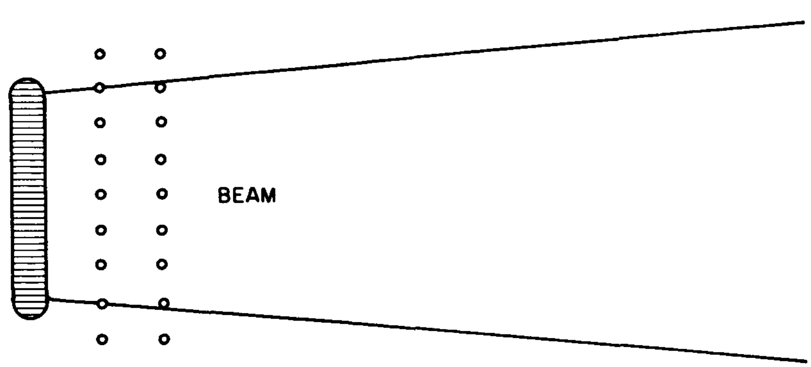

Fig. 1. Accelerator electrodes

Beam behavior cannot be tested on Earth in any complete sense, because no vacuum tank can have the electrical boundary conditions of outer space. If sputtering (erosion) problems prevent the use of grids in the final design, the equipotential surfaces will still function as virtual grids in the arrangement shown in Fig. 1.

Numerical estimates will be made for a beam that is typical of those planned for fairly advanced spacecraft. No particular vehicle or mission is implied, and the numbers are subject to considerable modification. The following values are assumed: 1-m^2 accelerator area, 60-amp ion current, 2000-ev ion energy (2000 volts emit to decel), and a 2000°K electron temperature. From these values, one can readily derive the ion velocity of 5×10^4 m/sec, mean electron velocity of 3×10^5 m/sec, electron-to-ion mean velocity ratio of 6, ion charge density of 1×10^{-3} coul/m^3, ion number density of $7 \times 10^{15} m^{-3}$, and mean electron energy of 0.3 ev or 4×10^{-20} joule. Mean free path will be discussed later. It turns out that collisions are not negligible.

To understand the charge distribution in the beam, consider what it looks like shortly after the ion motor of a space vehicle is turned on. The region occupied by the ions will be shaped somewhat like the conical region shown in Fig. 2. Assume

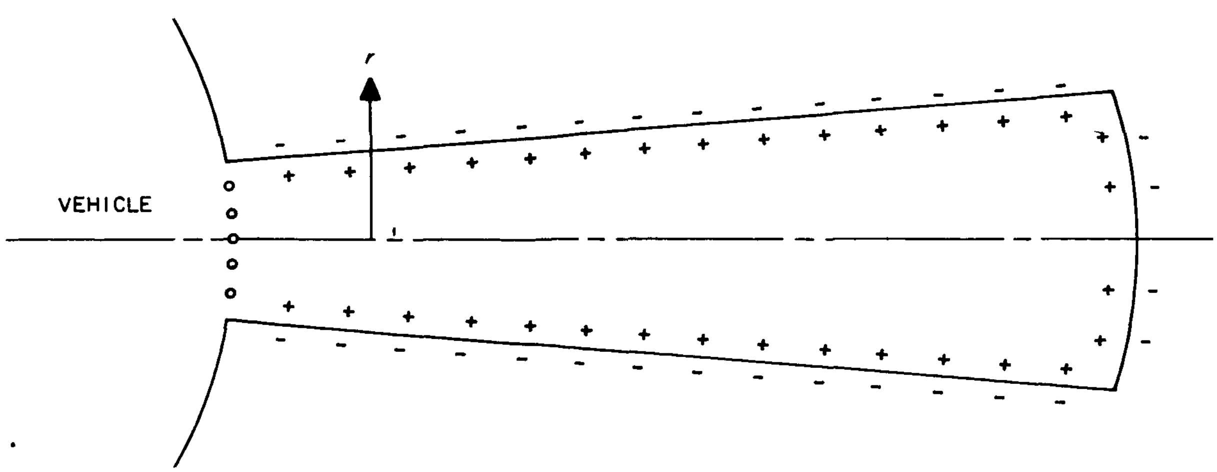

Fig. 2. Charge distribution in the beam

that the neutralizing electrons are emitted from hot filaments located well inside the ion beam. (Section III discusses the case in which sputtering requires that the filaments be located outside the beam.) The electrons boil off with an average velocity that exceeds the ion velocity by a factor of 6. The electrons coast past the ion region on all sides. This leads to a charge distribution indicated by the plus and minus signs in Fig. 2. One can integrate the electric field connecting these charges to find the electric potential V. Along r, a radial line from the thrust axis, V has the qualitative shape of Fig. 3. Since electrons are far more mobile than ions, the

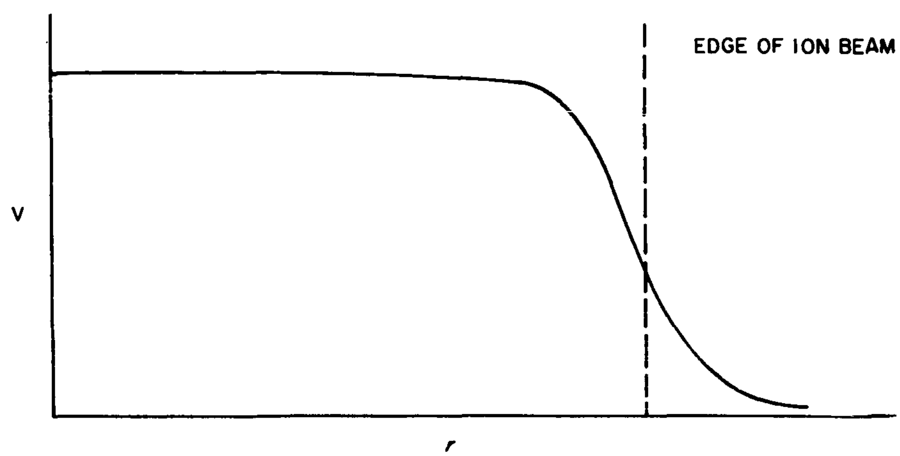

Fig. 3. Qualitative radial potential

acceleration of ions (outside the vehicle, not the main acceleration) is a minor perturbation in the electrical problem. Thus, the potential energy of the electrons, $U = -eV$, is a more convenient quantity than V, and the word "potential" will subsequently be applied to U rather than V. Inverting Fig. 3 gives for U the shape of a potential well that tends to contain the electrons (see Fig. 4). If the potential well is not deep enough to hold all the electrons, some escape, increasing the field and

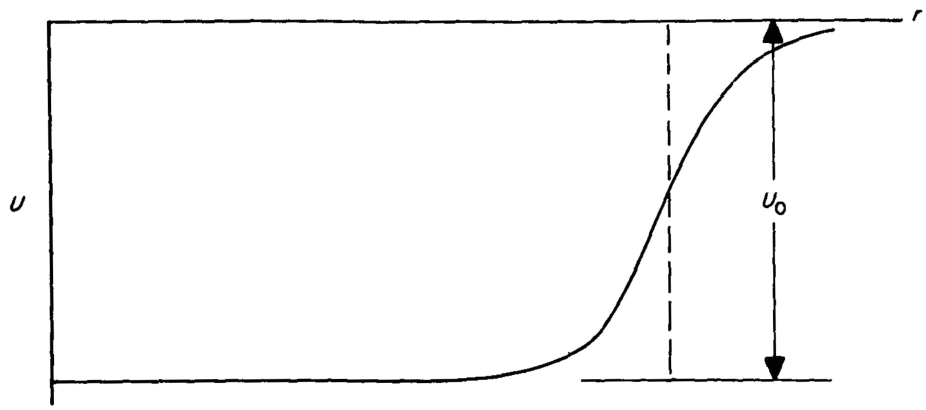

Fig. 4. Electron radial potential well

deepening the well, until finally it does contain them. The distance of electron overshooting will be estimated later. It is on the order of a fraction of a millimeter.

Strictly speaking, no finite potential can contain all the electrons on the tail of the velocity distribution function. Practical containment occurs when the number escaping is comparable to the number already drifting about in space. For example, space vehicles already in orbit have metallic parts that hold electrons by only a few volts (the thermionic work function). Occasionally an electron attains the energy to escape, yet no one worries seriously about these objects acquiring an ever-increasing positive charge.

The above description of electron containment in a potential well applies in that part of the beam that is well removed from the vehicle. Now consider electron containment near the vehicle. Looking from behind the vehicle directly into the electrostatic accelerator, electrons find a large potential barrier between the accel and decel grids which keeps them from escaping into the ion emitter. Thus, the electrons are completely trapped in the beam region, in which they accomplish permanent neutralization if they are prevented from escaping across the one area not yet considered: the periphery of the beam near the vehicle, where end effects due to the metallic structural parts become important. To close this escape route, the vehicle should be designed so that the electron emitter is biased positive with respect to the body of the vehicle by an amount U_0 sufficient to contain the electrons (a few volts, like a typical thermionic work function). Then the electrons are bound on the vehicle end of the beam region, as well as on all the open surfaces. Neutralization is accomplished by simply maintaining the electron current equal to the ion current. Pockets of positive or negative charge can exist in the interior of the beam only if there are not enough collisions to thoroughly randomize the electron velocity distribution. Later it is shown that there are plenty of collisions by a wide margin. In Fig. 5, the dashed lines indicate the bounds of the main potential slope. Here the decelerating grid is also biased to $+U_0$ and may itself be all or part of the electron emitter. If the emitter were not biased, metallic structural parts would be in the potential well, and capture of electrons by these parts might be a serious leakage of the neutralizing current.

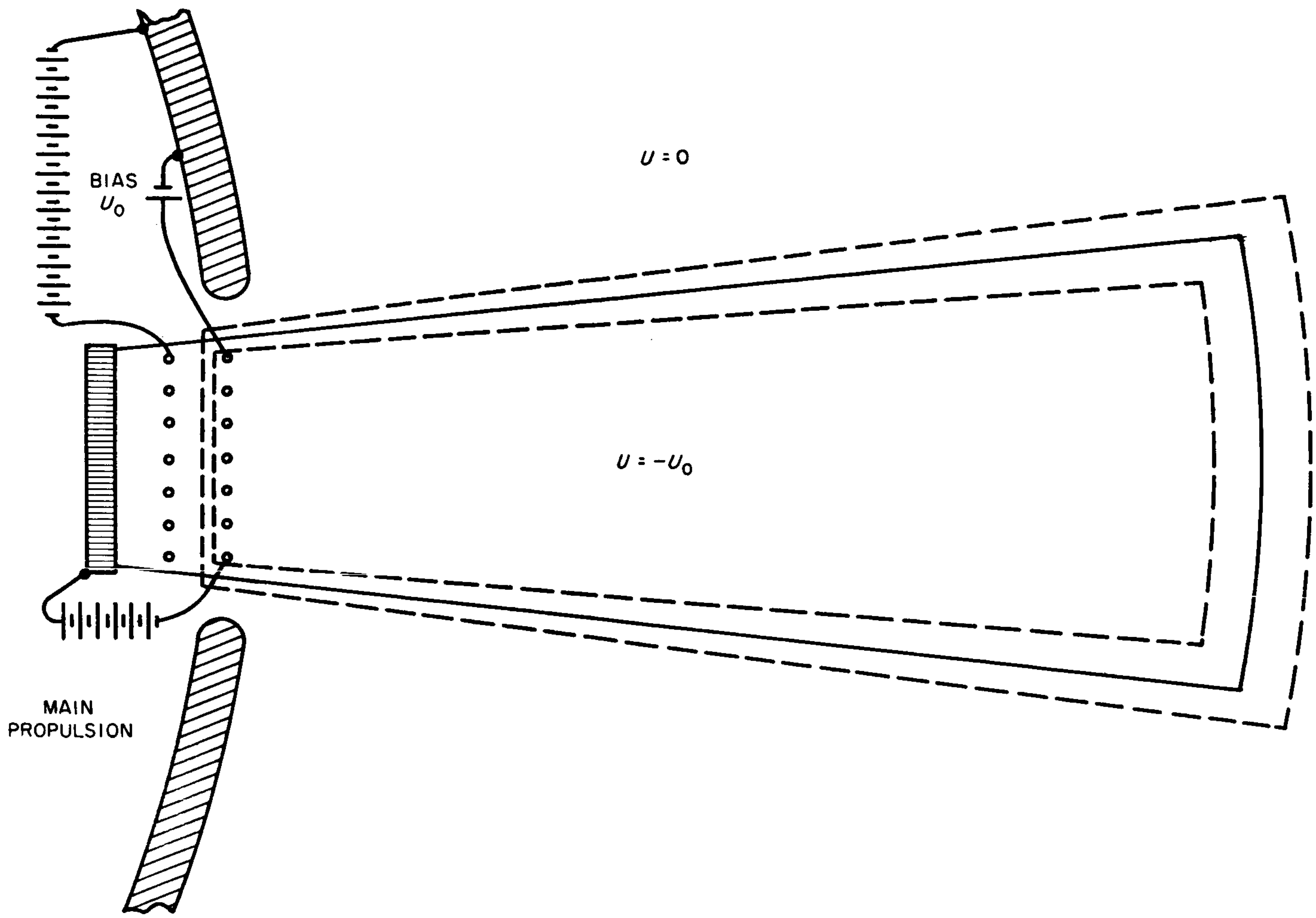

Fig. 5. Bounds of the complete potential well

The situation inside a neutralized beam can be likened in some respects to the inside of a hot chunk of solid metal. In solid metal the positive centers are rigidly fixed in a crystal lattice, while in the beam they are only relatively fixed by virtue of their greater mass. In both, the hot electrons overshoot the bounds of the positive charges and thus form a potential well. The electrons are contained in this well except for a negligible few on the extreme tail of the Fermi-Dirac distribution function. The beam density is great enough to randomize the electrons by collisions with ions. Hence, one would no more expect unneutralized charge to persist in the beam than one would expect it in a chunk of metal which had been positively charged and then neutralized by the addition of electrons.

The flow of electron "gas" into the beam can also be compared to the flow of a hot gas through an orifice into a cylinder with a piston expanding the cylinder and cooling the gas. The moving piston corresponds to the outer bound of the beam region moving at the velocity of the ions, and the cylinder to the side walls of the potential well. Eventually the "piston" will stop at a great distance from the vehicle. This happens when most of the electrons have had time to recombine with the ions to form neutral atoms, or otherwise to cool and intimately mix, removing the particles from the electrical problem.

To continue the piston analogy, one must imagine some mechanism that removes gas particles at random. Acceleration of the vehicle and radial acceleration of the ions also contribute to the expansion of the electron gas container. If the gas is collisionless and the cylinder and piston walls perfectly smooth, then the flow pattern might be such that two points in the cylinder have substantially different densities, analogous to imperfect neutralization. However, a very little sand suspended uniformly in the cylinder will nearly wipe out such inequalities, and is analogous to the effect of electron - ion collisions.

When the ion motor is first turned on, the ion density has sharp boundaries, and the walls of the potential well are fairly steep. When the system reaches equilibrium, the far boundary of the ion density will be many kilometers behind the vehicle, and the boundary will be so diffuse that it smears over a distance measured in kilometers. Thus, the far wall of the potential well will have an extremely gentle slope, but the height is undiminished. The electrons that do not fall into bound states of cesium are still totally reflected. The gentle slope merely means that the fast electrons coast much farther than the slow ones before reflection.

The outer potential wall reflects electrons continuously as it recedes from the vehicle and finally stops at a great distance. Hence, the back flow of electrons into the vehicle is uninterrupted from the moment the motor is turned on.

It was seen in Fig. 5 that, with proper bias, back flow is not lost at all, but is merely reflected at another wall of the potential well. Only back flow into filament wires is lost. For sufficiently high temperature, this flow is negligible compared to the flow out, because the electron density is so much greater inside the metal than out. If back flow into the filaments is not negligible, the loss of electrons merely lowers the beam potential relative to the filament until the loss is negligible. The required potential drop could not exceed the thermionic work function (a few volts). This would increase the confining potential by the same amount, but not the emitter bias.

Although the shape of the potential well is treated in a later section, it seems advisable here to estimate the width occupied by the main slope of the potential walls to see how this width compares with the dimensions of the vehicle. Close to the vehicle, assume a sharp boundary for the ion number density as given by ρ_+ in Fig. 6a.

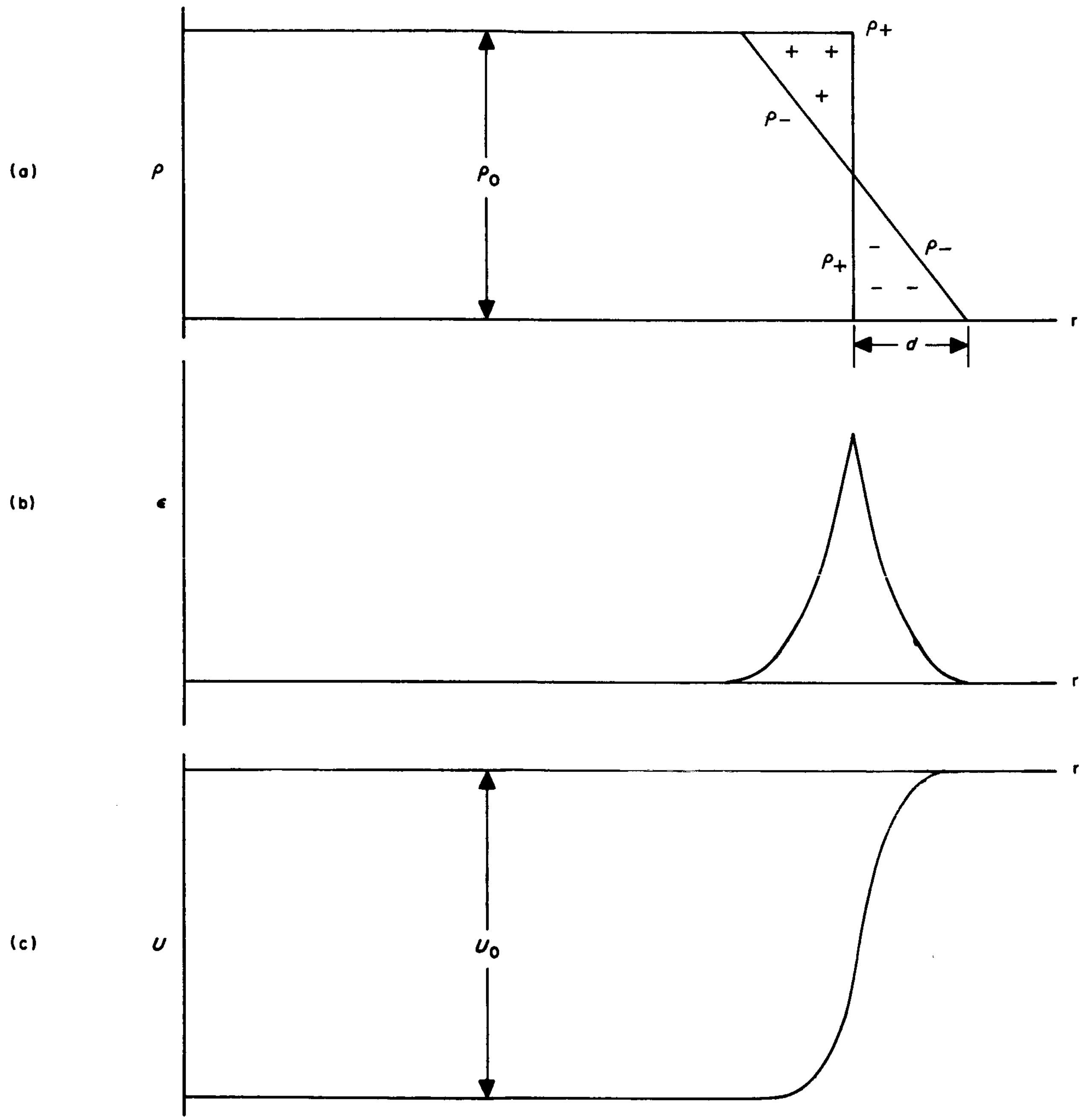

Fig. 6. Assumed charge density, electric field, and potential for estimate of distance electrons extend past the beam

For the electron density assume, for simplicity, a linear slope extending past the beam by a distance d to be calculated. Neglecting small effects from the divergence of the beam and the proximity of conductors, it can be said that the electric field is radial and can be calculated from Gauss's law. The intersecting parabolas of Fig. 6b result. Integration gives the potential, the intersecting cubics of Fig. 6c.

To check the assumed charge density for self-consistency, one should use statistical mechanics to solve for the density of hot electron gas in the potential of Fig. 6c to ascertain whether it gives back the density of Fig. 6a. Let us accept the shape for the present; improvement will come later. The numerical value of U_0 from the steps described above is

$$U_0 = \frac{e\rho\, d^2}{6\,\epsilon_0} \frac{\text{ev}}{\text{mm}^2} \tag{1}$$

Since potentials which confine the vast majority of electrons inside a hot metal are the order of a few volts, consider $U_0 = 2$ volt. Putting the previously assumed value of charge density, 10^{-3} coulombs/m^3, into Eq. (1) gives d = 0.3 mm, a dimension that is small compared with the structure in Figure (5).

III. ELECTRON EMISSION OUTSIDE THE BEAM

Suppose that it is desirable, because of sputtering or any other reason, to locate the electron emitter outside of the ion beam. In this case, pick an arbitrary point for emission such as P in Fig. 7. To understand the new charge and potential distributions, suppose that the emitter was originally in the beam, and then was

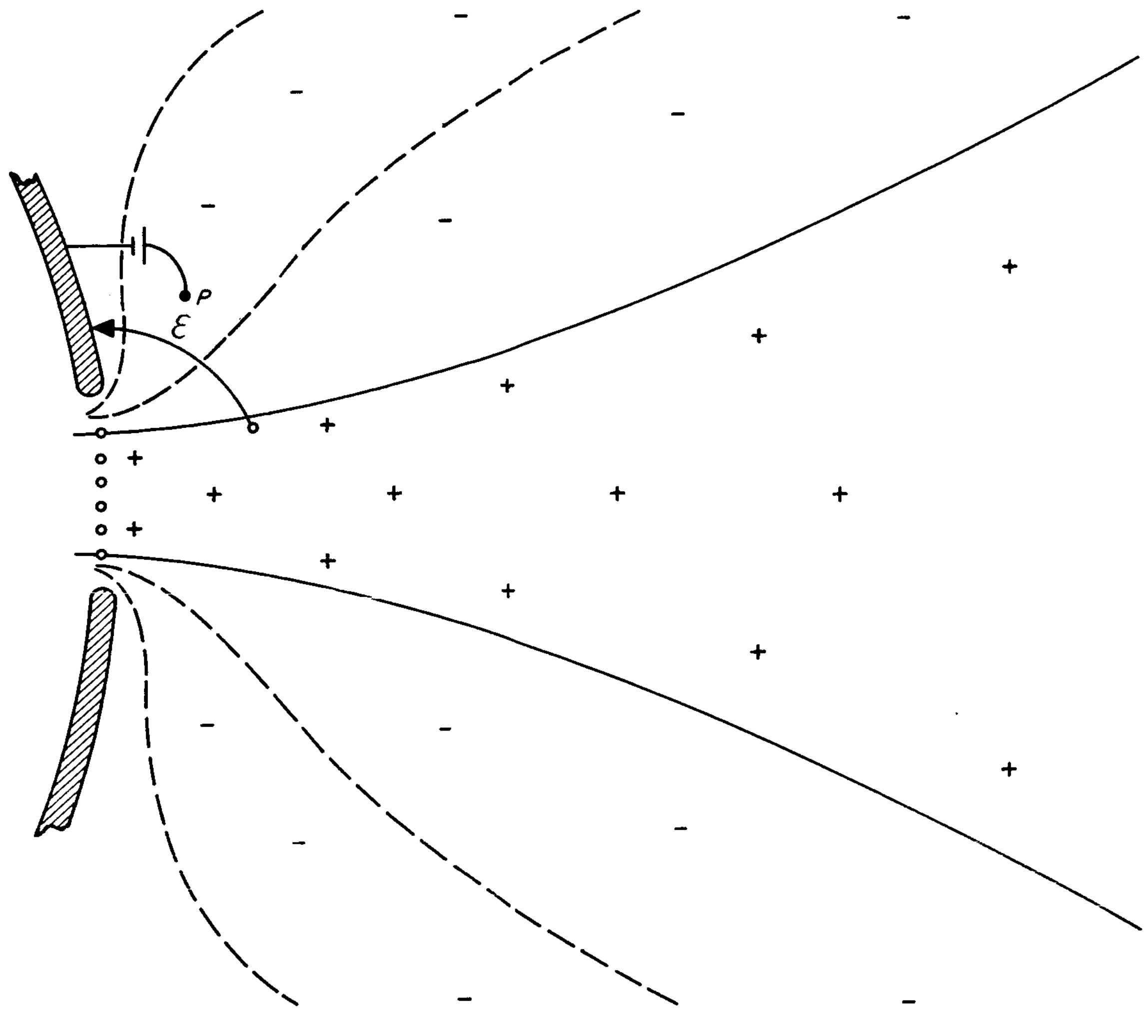

Fig. 7. Electron emission outside the beam

suddenly moved to P. Now consider the approach to the new equilibrium. The electrons which reach the beam do so by falling into the potential well and thus acquire enough kinetic energy to escape from the other side. Hence, neutralization is not accomplished, the beam becomes more positive, and the potential well becomes wider and deeper. This process continues until the well becomes so wide that it engulfs the source P to the depth U_0 that entraps the electrons. In this condition the new equilibrium is achieved. As before, the point P must be biased to $+U_0$ relative to the vehicle structure. Approximate equipotentials are dotted in Fig. 7. The potential is shaped qualitatively like Fig. 8 along a cross section. In accordance with Poisson's equation, the inflection point on the curve is at the edge of the beam.

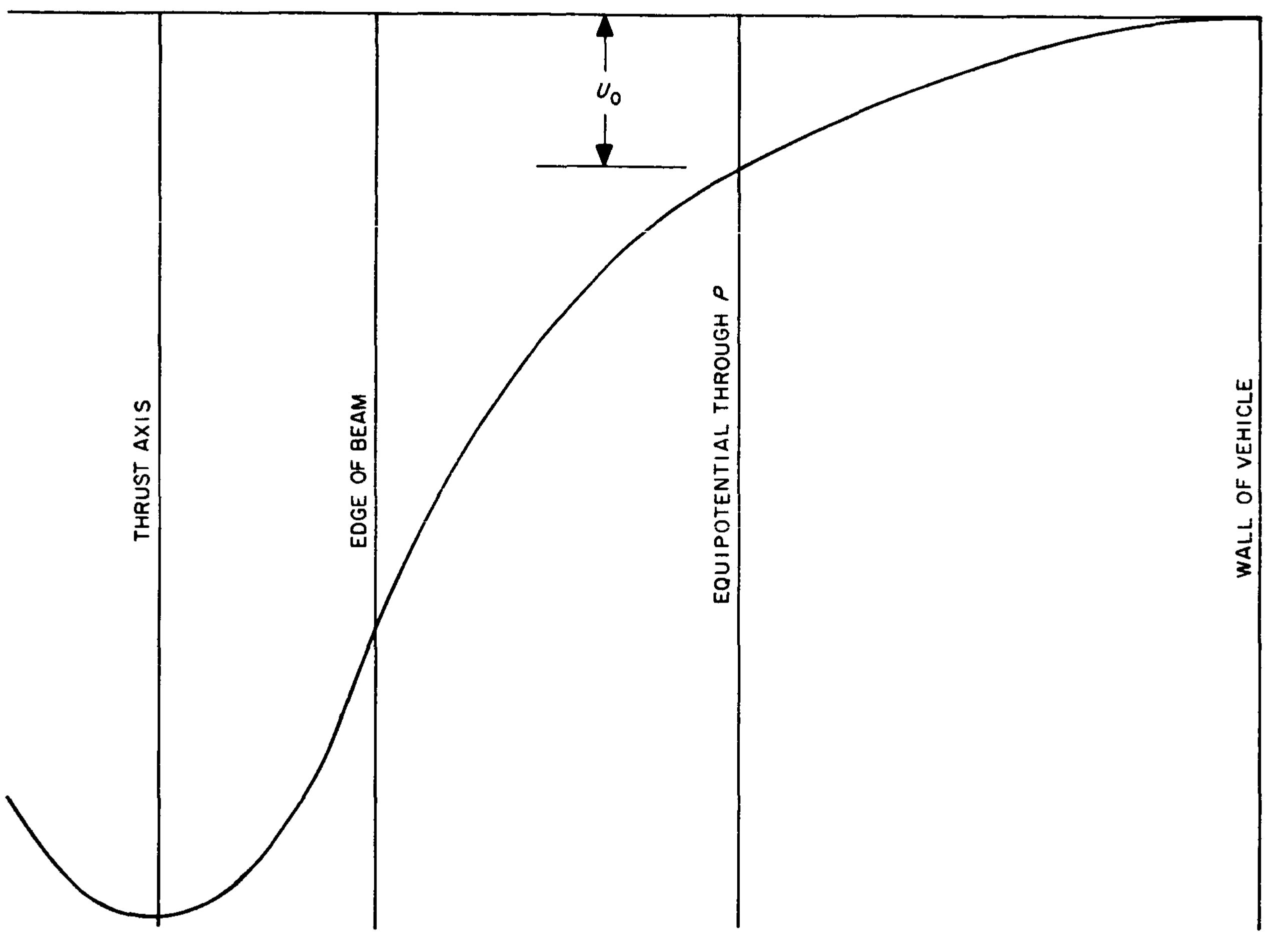

Fig. 8. Potential well with emission outside the beam

This case is unsatisfactory. Since equipotentials bend back toward the vehicle, lines of force such as that marked $\mathcal{E}$ in Fig. 7 connect to the ions and spread the beam. The wide dispersion of electric charge makes a wider cone angle of radio interference.

A fairly satisfactory solution to the problem is to inject the electrons from filaments that surround the beam as closely as possible without being in it. The beam is fairly narrow, and the equipotentials do not bend back toward the vehicle. The distance d of Eq. (1), the amount that the electron density overshoots the ion density, is approximately the distance of the filament from the beam. If this distance is 2 mm, then Eq. (1) gives 20 volts for the potential depth. As the electrons enter the beam from the side, they gain radial momentum falling through a 20-volt drop. Reflecting back and forth from side to side, they impart this momentum many times to the ions; that is, the 20-volt electrons cause an unfortunate pressure which tends to spread the beam slightly. This effect will be estimated in Section VIII.

To obtain the narrowest possible beam with a reasonable gap between the filaments and the high-speed ions, the ion motor could be designed with unaccelerated ions filling the gap. A very small cesium current could be emitted from a hot surface next to the electron emitter.

IV. COLLISIONS

The previous sections assumed that electrons fill and neutralize the interior of the beam region regardless of the details of electron injection. This implies that collisions are important in randomizing the electron motion. The purpose of this section is to justify that assumption. The concern will be only with electron-ion collisions. Only these collisions can change the mean velocity of a cloud of electrons. Electron-electron collisions can only randomize the relative velocities of electrons.

According to Spitzer (Ref. 1, chapter 5), the mean free path for encounters that deflect the electron 90 deg or more is

$$\lambda_{90} = 1/\pi\rho_i p_0^2 \tag{2}$$

where

$$p_0 = e^2/4\pi\epsilon_0 m_e v_e^2 \tag{3}$$

ρ_i is the ion number density, and v_e the electron velocity. For the values of ρ_i and v_e assumed in this report, one obtains λ_{90} = 6.4 meters.[2]

For long-range coulomb forces, it turns out that the effects of numerous distance encounters are much more important than the few close encounters. Spitzer defines a diffusion time (hence a path λ_D) during which the average electron is turned

[2]Strictly, this is not the mean free path of the electron gas, but rather the mean free path of those electrons in the gas that have mean velocity. Since $\lambda_{90} \sim V_e^4$, the low end of a Maxwellian distribution would contribute many electrons with extremely short λ_{90}.

in a random direction through 90 deg as a result of encounters of all types. This is given by

$$\lambda_D = \lambda_{90}/8 \ln \Lambda \tag{4}$$

where

$$\Lambda = \frac{12\pi}{\rho_e^{1/2}} \left(\frac{kT\epsilon_0}{e^2} \right)^{3/2} \tag{5}$$

and (3/2)kT replaces (1/2) $m_e v_e^2$. For the assumed values, $\ln \Lambda = 9.5$ and $\lambda_D = 8.4$ cm. This distance is small compared to the dimensions of the beam.

It should be noted that in calculating λ_D thermal velocity was assumed for the electrons. There is a chance that in the final design the electrons will fall into the beam through an accelerating potential. This, of course, lengthens λ_{90} and λ_D, because in each encounter a fast electron does not stay near the ion as long and so does not receive as much impulse. To correct for this, note that λ_{90} is proportional to the square of the kinetic energy; λ_D is nearly proportional to the square ($\ln \Lambda$ holds it down a bit), and (3/2)kT corresponds to 0.26 ev. For a large, 20-volt acceleration,

$$\lambda_D = \frac{6.4 \text{ m} \times (20/0.26)^2}{8(9.5 + 3/2 \ln 20/0.26)} = 300 \text{ m}$$

This is still short compared to the length of the beam.

To further discuss the role of collisions in the neutralization problem, it is convenient to think of the electron density in two distinct parts. Electrons of the first part will be called "fresh" electrons, those which have not suffered a large angle collision since their emission. The remainder will be called collision electrons.

The important point for the ion propulsion problem is this: No matter how long the mean free path may be, collision electrons make up the greater part of the electron density in all regions of the beam. This fact will be demonstrated in two parts. First, it will be shown that collision electrons dominate the beam as a whole. Then it will be shown that collision electrons dominate next to the vehicle (emitter), where one might expect the fresh electrons to dominate. (It is assumed that one electron is emitted for each ion, i.e., electron flow is not space charge limited.)

To show that collision electrons dominate the beam as a whole, note that an electron can only escape the electric potential well by recombining with an ion to form a neutral atom (neglecting the very few electrons on the extreme tail of the velocity distribution function). Thus, the mean ratio of recombination time to collision time is a measure of the predominance of collision electrons. This ratio is much greater than 1, since more and/or closer encounters are required to radiate energy than merely to randomize the velocity.

To estimate the order of magnitude of this ratio, consider the recombination equation (Ref. 2)

$$\frac{dn}{dt} = -\alpha n^2$$

The recombination coefficient α is experimentally[3] found to be on the order of

[3]The experimental value is taken at much higher pressure than that existing in an ion beam and may not be constant in the extrapolation because of complicated processes at high pressure. The theoretical value of α is lower by orders of magnitude. The lower value would strengthen the argument to follow. Therefore, use of 4×10^{-10} leads to a conservative estimate.

4×10^{-10} cm^3/sec. A recombination time can be defined as

$$\tau_R = \frac{n}{-dn/dt} = \frac{1}{\alpha n} = 0.4 \text{ sec}$$

Collision times are on the order of 3×10^{-5} sec; thus the ratio is on the order of 10^4.

One might suppose that fresh electrons would dominate within a distance of the vehicle much less than a mean free path, even though the collision electrons dominate the beam as a whole. This is not the case. The collision electrons diffuse upstream toward the vehicle as well as downstream. To estimate the charge density caused by diffusion upstream, it is necessary to consider the relation between current (j) and charge density ($\pm e\rho$) for the three types of particles, i.e., ions, fresh electrons, and collision electrons. These relations are

$$\vec{j}_i = \vec{v}_i \, e\rho_i$$

$$\vec{j}_{fe} = -\vec{v}_{fe} \, e\rho_{fe}$$

$$\vec{j}_{ce} = -\vec{v}_{ce} \, e\rho_{ce}$$

where the $\vec{v}$'s are mean velocities, and the subscripts refer to the type of particle. In the steady state $\vec{j}_i + \vec{j}_{ce} + \vec{j}_{fe} = \vec{j}_{total} = 0$. Also, next to the vehicle, $\vec{v}_{ce} = \vec{j}_{ce} = 0$, since diffusion toward the vehicle balances diffusion away. In the above equations this implies that

$$\frac{\rho_{fe}}{\rho_i} = \frac{v_i}{v_{fe}}$$

This ratio is no greater than 1/3, since the mean thermal velocity of fresh electrons exceeds the ion velocity by at least a factor of 3, depending on the temperature and potential of the emitter.[4] Thus, by outrunning the ions fresh electrons accomplish at most 1/3 of the neutralization, and collision electrons are actually sucked upstream by coulomb attraction to the remaining 2/3 or more of the ion density. If approximately complete neutralization is achieved, the collision electrons will constitute about 2/3 or more of the electron density next to the vehicle.

To show that the collision electrons very nearly complete the neutralization of the remaining 2/3 or more of the ions, it is necessary to consider the fact that the collision electrons tend to diffuse preferentially downstream because of the drag of the ions. Consider a coordinate system in which the ions are at rest. A stationary electron density appears to be moving in this system. Thus the collision electrons near the vehicle result in 40 - 60 amperes of upstream current in the moving system. Applying Ohm's law (Ref. 1, Eqs. 5-37)

$$\mathcal{E} = j\eta$$

$$\eta = 6.5 \times 10^3 \frac{\ln \Lambda}{T^{3/2}} \text{ ohm cm deg}^{3/2}$$

indicates that the drag will support an electric field of 0.4 volt/meter, where T is assumed to be 2000°K, and j = 60 amp/meter^2. This is a measure of how close the collision electrons will come to completing the neutralization of ions near the vehicle in the steady state. Assuming that equal numbers of ions and electrons have been

[4]This is the ratio of the mean vector velocities. The ratio of mean scalar velocities is more like 6.

One might suppose that v_{fe} is reduced by electric fields in the plasma close to the vehicle, but the conductivity of the plasma will not support such fields, as is shown very shortly.

emitted (so that the vehicle is not a source of electric field) the residual positive space charge near the vehicle can only be that amount which gives rise to 0.4 volt/meter of electric field. This is a negligible deviation from total neutralization.

Several papers have appeared in the literature treating the neutralization problem as though all electrons were fresh. From the above considerations it appears that these treatments have little validity. Certainly they present much too pessimistic a view of the difficulties involved in achieving complete neutralization.

V. TANK TEST

It is difficult to design a tank test of an ion propulsion motor that adequately imitates space environment. Certainly the vacuum must be very good, so that artificial neutralization does not result from ionized air. At the place where the beam strikes the tank, a grid should be arranged to reflect electrons in the same manner as the potential wall reflects electrons in space. Figure 9 shows this arrangement. Note that the motor is insulated from the tank and provided with a floating power supply.

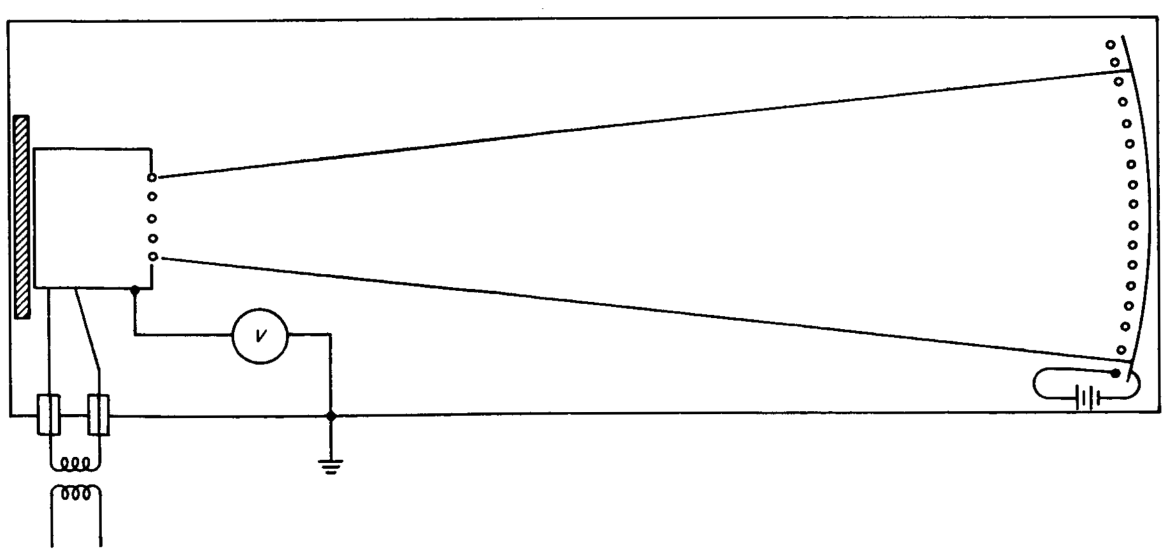

Fig. 9. Tank experiment

The grid shown in Fig. 9 must not reflect all the electron current. Unlike the space environment, the tank does not allow time for cooling and recombination, and a quantity of electrons must be let through the grid to imitate recombination artificially. The best imitation of space is obtained if the partial reflection of electrons is not too velocity-selective. Holes in the grid might be used for partial reflection, but rapid pulsing of the grid bias provides more ease of control. The reflected electrons would be bunched in pulses; but if the pulse frequency is sufficiently high, the spread in the velocity distribution would damp out the pulses before the electrons travel far upstream.

Unlike the condition in space, it is possible in this test arrangement for electrons to escape without ever colliding with an ion. Depending on the mean free path, it might be desirable to provide some artificial randomization of the electron orbits. This randomization is automatically accomplished if the grid wires are widely spaced and irregular; this spacing gives a diffuse character to the reflecting surface. The diffuseness can be enhanced or controlled by connecting alternate grid wires to different pulse sources, which may or may not apply different waveforms or phase angle in the pulses of bias. Thus there can be an electric field component between wires that gives the electrons a sideways component of impulse.

VI. STABILITY

Happily, the neutralization problem has self-regulating characteristics. If the electron emission current deviates from the ion current, then the electric fields that develop tend to correct the condition. Even the IR drop in the electron supply tends to adjust the bias if it should drop too low and allow electrons to escape to the vehicle structure.

Suppose the entire electric circuit of the propulsion system is floated with respect to the vehicle structure or, at most, connected through a high-resistance leak R, as shown in Fig. 10. Then the electrons which escape from the potential well to the vehicle across the gap G will self-bias the emitter relative to the vehicle.

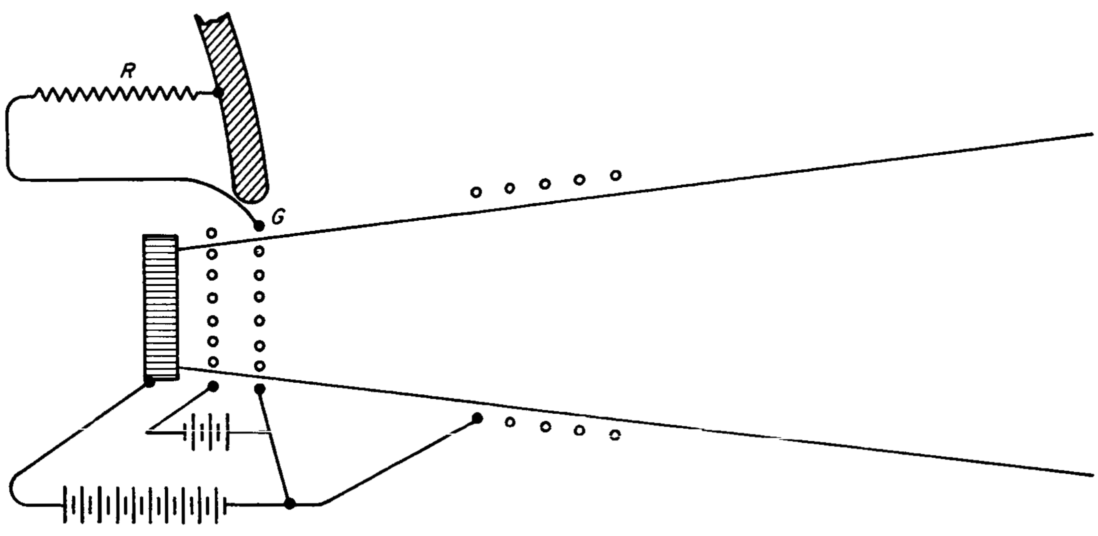

Fig. 10. Self bias

VII. ELECTRON DISTRIBUTION

To find the ion and electron distributions rigorously would require solving the Boltzmann equation which includes the collision term. Such a task is beyond the scope of this report. However, a rough approximation will be obtained by using the equations .

$$\Delta^2 V = e(\rho_e - \rho_i)/\epsilon_0 \tag{6}$$

and

$$\rho_e = \rho_0 \exp\left[eV/kT\right] \tag{7}$$

Here, ρ_i and ρ_e are the number densities of ions and electrons, respectively. Equation (6) is Poisson's equation. Equation (7) is the spatial distribution function that goes with Maxwellian velocity distribution to form the thermal equilibrium solution to the Boltzmann equation. Equation (7) applies only as a rough approximation because the fresh electrons have not reached thermal equilibrium.

For the ions, the collisionless Boltzmann equation suffices, because electron-ion collisions have little effect on the massive ions, and ion-ion collisions are not important because the initial velocity of the ions is so nearly uniform.

In this section the radial electron density is estimated in a cross-sectional plane close enough to the vehicle to allow the ion orbits to be approximated by straight lines, and far enough from structural parts to neglect end effects. In other words, ρ_i is taken to be a constant ρ_0 inside an infinite cylinder of radius a, and zero elsewhere. Elimination of V between Eqs. (6) and (7) gives

$$\frac{d}{dr}\left(r\frac{d}{dr}\ln f(r)\right) = r\left(\frac{f(r) - 1(a-r)}{h^2}\right) \tag{8}$$

where $f(r) = \rho_e(r)/\rho_0$, 1 is the unit step function, and h is the Debye length $h^2 = kT\epsilon_0/\rho_0 e^2$. For a 2000°K, plasma, h = 0.026 mm.

Equation (1) and the discussion following it have already shown that the main change in ρ_e occurs near the beam boundary and occupies a distance very short compared to the radius. Consequently, little error is introduced by replacing r with a where it multiplies other functions in Eq. (8). This gives

$$\frac{d^2}{dr^2}\ln f(r) = \frac{f - 1(a-r)}{h^2} \tag{9}$$

The solution with the correct boundary conditions is given for $r > a$ by

$$f(r) = 2\left(\frac{h}{r - b}\right)^2, \quad r > a \tag{10}$$

where $b = a - \sqrt{2eh}$, and e is the numerical constant, not the electronic charge. For $r < a$, the solution is

$$\frac{a - r}{h} = \sqrt{\frac{1}{2}} \int_{-\ln f(r)}^{1} f(r) \frac{ds}{\sqrt{s - 1 + e^{-s}}}, \quad r < a \tag{11}$$

The outside ($r > a$) solution is readily verified by substitution. To obtain the implicit form, Eq. (11), for the inside solution, put $g(r) = -\ln f$ and $p = dg/dr$. Then Eq. (9) becomes

$$p \frac{dp}{dg} = \frac{1 - e^{-g}}{h^2}$$

which integrates to

$$\frac{p^2}{2} = \frac{e^{-g} + g - 1}{h^2}$$

or

$$\frac{dr}{h} = \sqrt{\frac{1}{2}} \frac{dg}{\sqrt{g - 1 + e^{-g}}}$$

The last form is the derivative of Eq. (11). Note that $f(a) = e^{-1} = 0.368$, and $hf'(a) = (2/e^3)^{1/2} = 0.316$. Figure 11 is a plot of f(r), with r measured from a in multiples of h. Equation (7) gives

$$V = \frac{kT}{e} \ln f \tag{12}$$

Figure 12 is a plot of ln f, i.e., voltage in units of kT/e. For 2000°K, kT/e = 0.172 volt.

The discussion of Eq. (1) gave a preliminary estimate of 2 volts for the potential well and 0.3 mm for the distance that the electron density overshoots the beam. On the scale of Fig. 11 and 12, these numbers become

$$\Delta\left(\frac{V}{kT/e}\right) = 12 \quad \Delta\left(\frac{r}{h}\right) = 12$$

The previous conclusion, namely that electrons are confined within distances small compared to other dimensions, is confirmed here.

VIII. BEAM SPREADING

Figure 12 indicates that the ions at the edge of the beam are on a potential slope which tends to spread the beam. It is important to estimate this spread because the plasma has an index of refraction and will interfere with radio communication if the beam spreads so wide that it intersects the antenna pattern with an appreciable density. Figure 12 indicates that the electric field reaches strongly into the beam a distance of 1 or 2 h, and then fades out rapidly in the next few h.

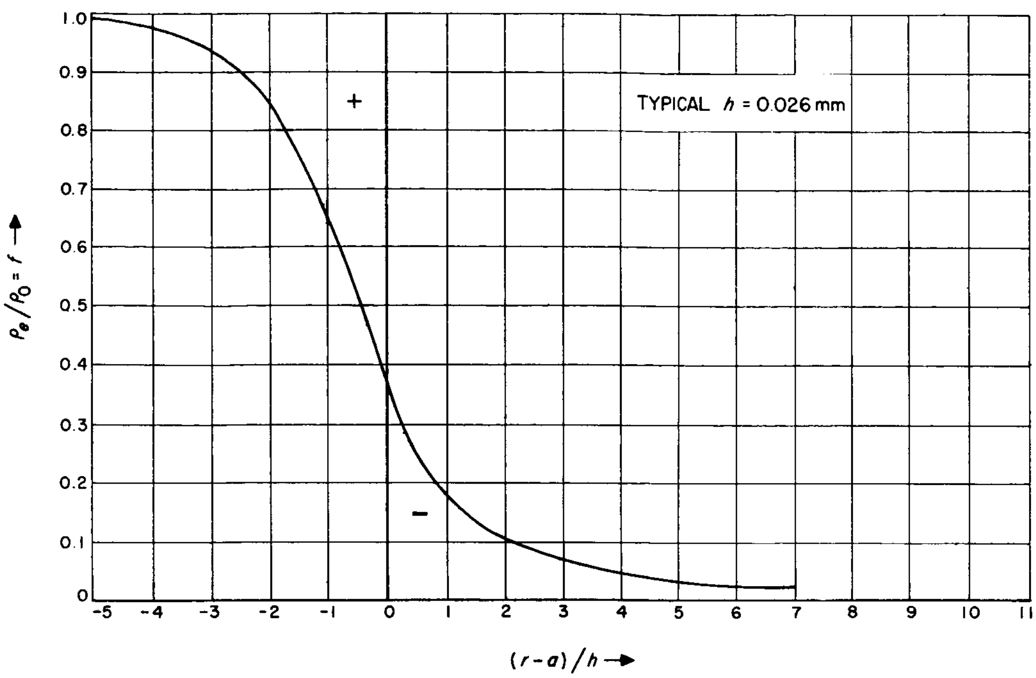

Fig. 11. Charge density as a function of radius in units of h, the Debye length

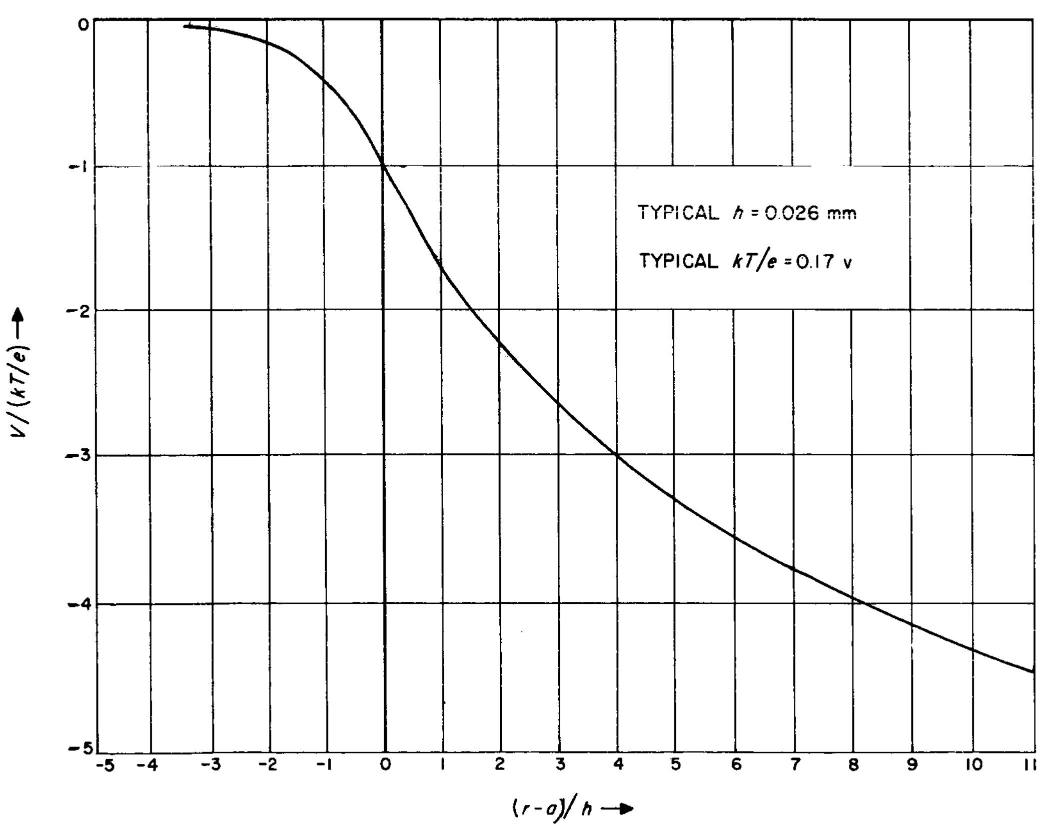

Fig. 12. Radial potential in units of kT/e

Thus, after the ions travel some distance from the vehicle, a layer of ions about 2h deep is peeled off the side of beam. With this layer out of the way, distributions such as shown in Fig. 11 and 12 apply to the remaining beam, and the next layer begins to peel off, etc. The reference to discreteness of layers is, of course, merely an explanatory device; the edge of the beam which is sharp near the vehicle becomes more and more diffuse, until finally the radial ion and electron densities have a smooth form qualitatively as shown in Fig. 13. The electron density still overshoots the ion density because of the electrons' great thermal velocity.

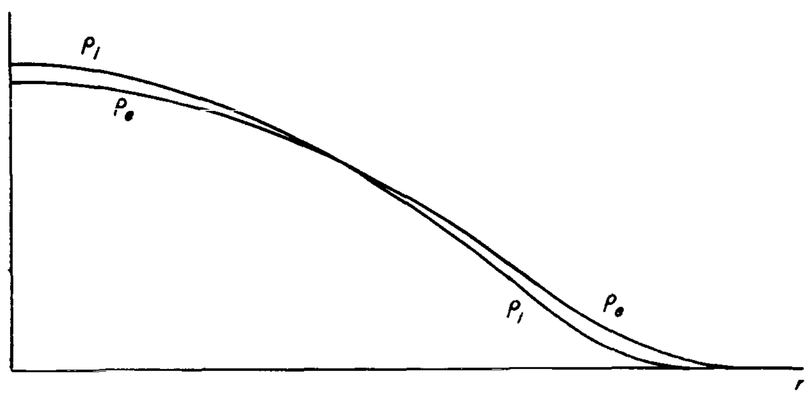

Fig. 13. Shape of density functions far downstream

This section will treat the topic in a way that is valid only for an approximate conception of the final cone angle of the beam. The ion orbits are assumed to be perfectly ordered, i.e., nonintersecting, and functions only of the initial radius. In the real case, ion orbits crisscross somewhat. The peeling described above will be regarded as a transient of no interest as far as finding the final cone angle is concerned. It is assumed that the radial ion density has already reached a shape somewhat like that of Fig. 14 for the initial condition at z = 0. For this discussion, it is assumed that as one follows the ions this shape persists while the ions spread out. The radial coordinates of the orbits are assumed to be proportional to a standard orbit, the orbit of one of the outer ions. That is,

$$r(z) = \frac{r_0}{R_0} R(z) \tag{13}$$

where z is the distance behind the vehicle, r is the radius from the thrust axis of an ion initially at r_0, and R is the radius of the standard orbit (see Fig. 14).

The easiest way to make the calculation is to find the pressure exerted on the potential walls by the electron gas inside, and then ascertain to what extent the walls yield to this pressure by associating with them the mass of the ions. For this purpose, the rarified electron gas will be regarded as a perfect gas. Clearly, the ion velocity vector is always parallel to the potential wall, and thus is perpendicular to the pressure and acceleration vectors. Hence the pressure only changes the ions' direction; it cannot influence their kinetic energy. If s is the distance along an orbit, this means that

$$s = v_0 (t - t_0) \tag{14}$$

is a partial description of the ion orbit.

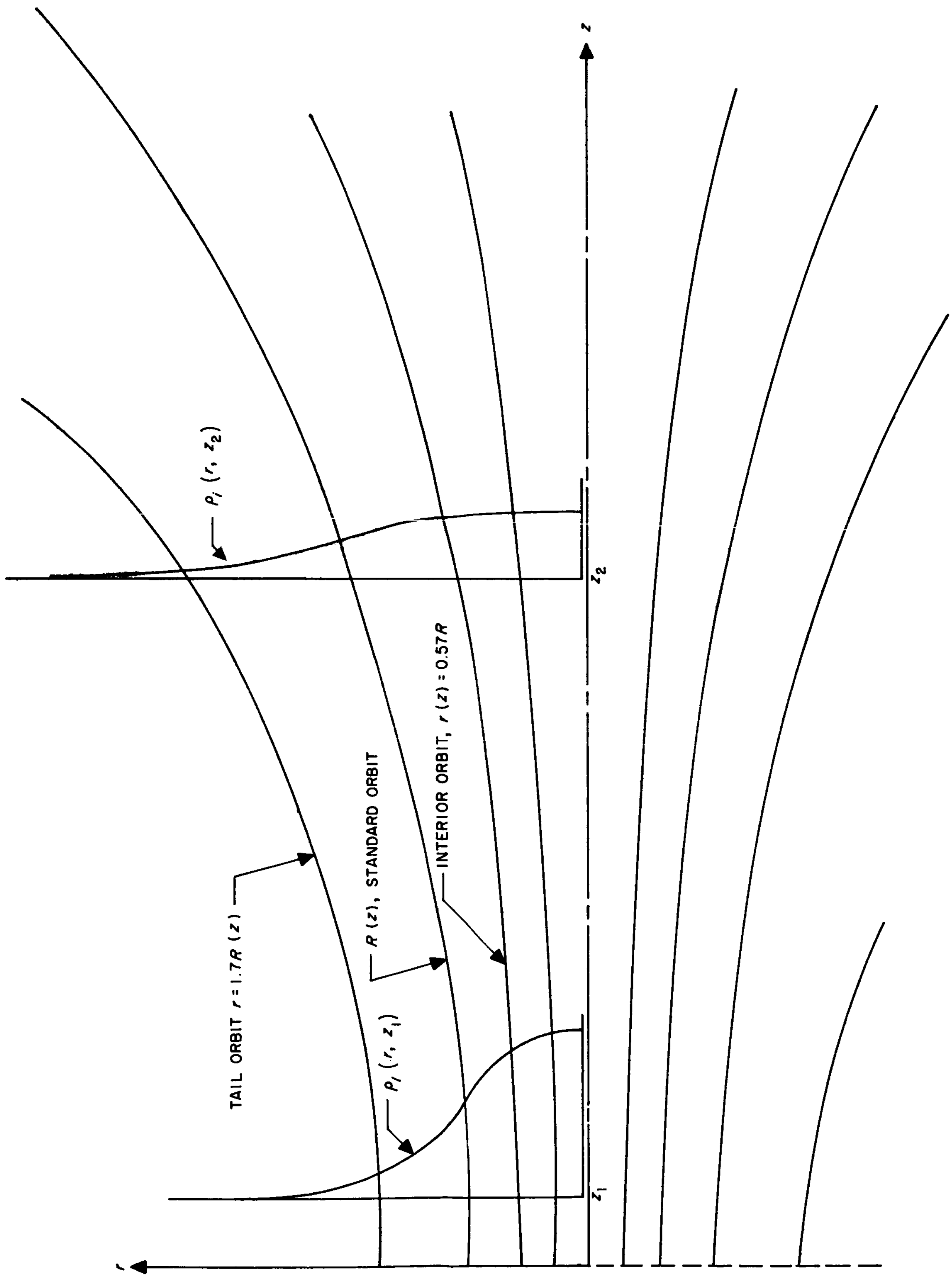

Fig. 14. Spreading ion orbits

It will be seen that the beam diverges so little that no distinction is necessary between s and the ordinary cylindrical coordinate z. For small divergence, the two are sensibly equal. Similarly r is nearly equal to distance measured from the thrust axis along a curve normal to the orbits. From Eq. (13) and (14),

$$z = v_0 (t - t_0), \quad \frac{dr}{dt} = \frac{dr}{dz} v_0$$

and

$$\dot{r}(t) = \frac{r_0}{R_0} \dot{R}(t) \tag{15}$$

Now consider the increment of beam from z to z + dz. For the ions in this increment,

$$\begin{aligned}(\text{d force}) &= (\text{acceleration})\ (\text{d mass}) \\ &= (\text{pressure})\ (\text{d area}) = (P)\ (2\pi R\, dz) \\ &= (kT\ \rho)\ \left[2\pi R\ (z)\ dz\right]\end{aligned}$$

Since the potential wall is actually smeared out over the main slope of the radial distribution function (Fig. 13), the above really defines the standard orbit R(z) as the effective average radius for computing d area of the electron reflecting wall. Similarly, ρ is an effective average number density for the perfect gas law.

The acceleration must be averaged over the cross-section of the beam, and then expressed in terms of the standard orbit acceleration, $\ddot{R}(t) = v_0^2 R''(z)$. From Eq. (15), $\ddot{r} = \ddot{R}\ (r_0/R_0)$. For this rough estimate, it will suffice to say that the ions which have $r > R$ just compensate the drop in $\rho_1(r)$ for $r < R$. The average r_0/R_0 is taken as though the density were constant for r <R(z), then zero for r(z)> R(z). This gives

$$\overline{\ddot{r}}(t) = \ddot{R}\ \overline{(r/R)} = \frac{2}{3}\ddot{R}(t)$$

$$\overline{r''(z)} = \frac{2}{3} R''(z)$$

Similarly, d mass = $m_{C_s} \rho\pi R^2\, dz$. Collecting the above force equations in terms of R(z) gives

$$R''\, R = 2\gamma^2$$

where

$$\gamma^2 = \frac{3\ kT}{2m_{C_s} V_0^2}$$

The solution of this differential equation is

$$\tilde{z} = \int_0^{\sqrt{\ln \tilde{r}}} e^{s^2}\, ds \qquad (16)$$

where $\tilde{z}$ and $\tilde{r}$ are dimensionless cylindrical coordinates defined by

$$\tilde{z} = \gamma \frac{z - z_0}{R_0}, \quad \tilde{r} = \frac{R}{R_0}$$

The constants of integration R_0 and z_0 appear in such a way that one may choose any distance scale factor (R_0) and any origin in the z direction (z_0).

Equation (16) is plotted in Fig. (15) for $\tilde{z} > o$. For $\tilde{z} < o$, note that $\tilde{r}$ is an even function of $\tilde{z}$. (Take the negative branch of the square root in Eq. (16)). From Fig 15 it would appear that $\tilde{r}(\tilde{z})$ approaches an asymptotic slope of 3.84, but actually

$$\frac{d\tilde{r}}{d\tilde{z}} = 2\sqrt{\ln \tilde{r}} \qquad (17)$$

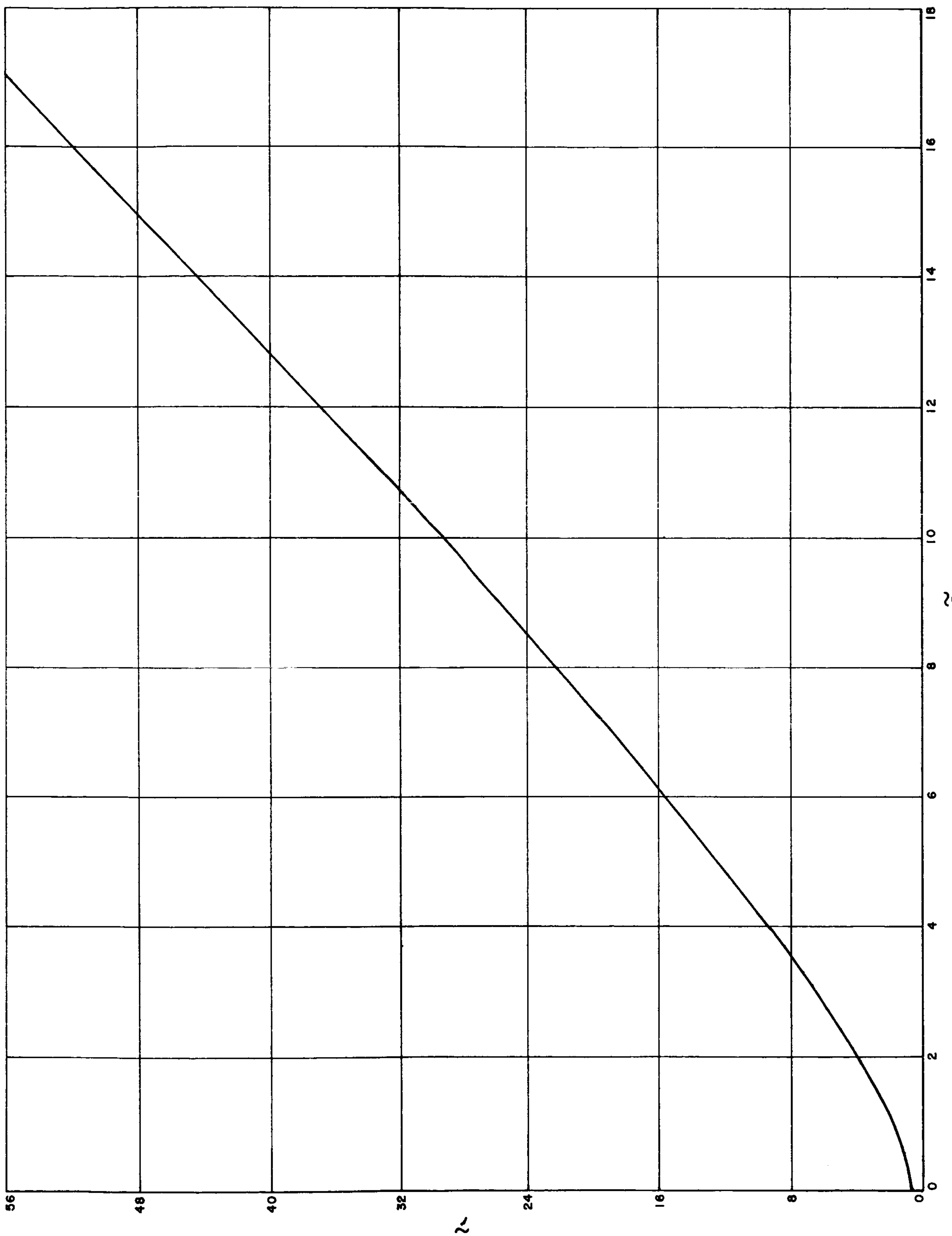

Fig. 15. Beam spreading function

diverges slowly. However, the divergence is so slow that in practical distances, before the beam has time to cool (by emission of bremsstrahlung), one may regard 3.84 as an asymptote. In real space the slope is $dR/dz = \gamma d\tilde{r}/d\tilde{z} \rightarrow 3.84\gamma$. For $T = 2000°K$, $m_{Cs} v_0^2/2 = 2000$ ev; the value of γ is 8×10^{-3}; and $dR/dz \rightarrow 3.1 \times 10^{-2}$. That is, the half cone angle of divergence is asymptotically 1.8 deg. If R_0 is taken as one meter, and $z_0 = 0$, corresponding to ions initially parallel, then the point $\tilde{z} = 18$, $\tilde{r} = 60$ on Fig. 16 is in real space $z = 2.25$ km, $R = 60$ m, $Z/R = 37$.

In interpreting these results, it must be remembered that R represents an effective radius in the radial distribution. Ions on the extreme tail of the distribution diverge at larger angles, perhaps double or more, say 4 or 5 deg.

Again consider the first layer of ions to peel off. If the potential of Fig. 13 did not change as a function of z, then the outer ion layer would fall through about one volt and diverge at an angle of

$$\left(\frac{1 \text{ volt}}{2000 \text{ volts}}\right)^{1/2} \text{rad} = 1.3 \text{ deg}$$

But since the potential spreads with the beam and follows the outer layer outward, it is reasonable that the final divergence of that layer should be 3 or 4 times this amount.

Also it should be noted that a beam which is initially diverging more than 1.8 deg corresponds to a portion of the r vs z curve far to the right of that shown in Fig. 16; i.e., z_0 is a large negative number, and the origin of z is shifted off the scale. It is clear from the slow rate of increase of slope in Fig. 16 and Eq. (17) that, in this case, the slope is sensibly constant at the initial value.

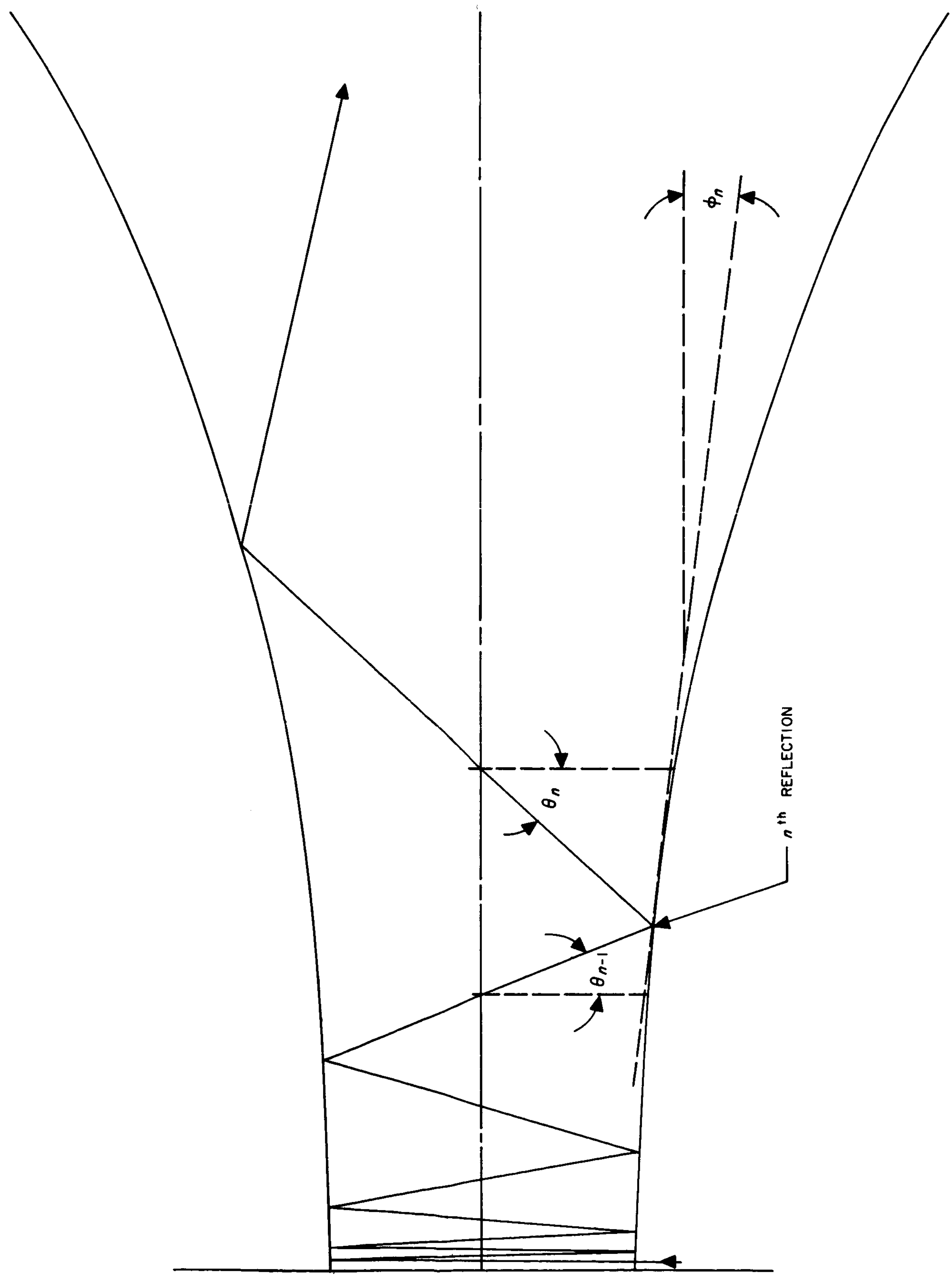

Fig. 16. Reflections of a high speed electron

Another case deserves special attention. It is the case in which the electrons fall into the beam sideways through an accelerating potential. Consider a large value of 20 volts. The mean free path is then very long, and an electron orbit looks like that of Fig. 16. In this case, the electron reflections pull outward quite strongly on the outer layer of ions.

The electrons undergoing their nth reflection bend the layer of ions which absorb the recoil through an angle

$$\phi_n - \phi_{n-1} = 2\,(p_e/p_i)\cos\theta_n \tag{18}$$

where p_e is the momentum of an electron, and p_i is the momentum of an ion times the fraction of the ions in the layer. The difference between $\cos\theta_n$ and $\cos\theta_{n-1}$ has been neglected in this expression. The initial electron fall into the beam deflects the layer by $p_e/p_i \equiv \beta$.

A rough estimate of the layer thickness can be calculated like a Debye length, but with the 20 ev energy replacing (1/2) kT in the Debye formula. This gives a thickness of 5.6×10^{-2} cm and $\beta = 0.09 = 5$ deg. For this large β, there are only 3 reflections of each electron, which can easily be traced graphically. This leads to a final ion layer deflection of 25 deg. It must be remembered that the discreteness of the layer is not to be taken very seriously. The thickness merely gives an idea of the number of ions which deviate markedly from the main beam, and the 25 deg is only an estimate of the average deflection in this group.

To summarize, it appears that the divergence of the main beam can be held down to 2 or 3 deg if it is initially within these limits as the ions flow from the accelerator. However, the pressure of hot electron gas can peel off a thin outer layer (1 or 2 Debye lengths thick: less than 1 mm) of ions and disperse them through cone angles which may exceed 25 deg if the electrons are energetic.

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